THE QUADRILINEAR FORM (1, 1, 1, 2)

BY

CHARLES FREDERICK STROBEL

A.B., University of Buffalo, 1935
A.M., University of Buffalo, 1937

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS IN THE GRADUATE SCHOOL OF THE UNIVERSITY OF ILLINOIS, 1941

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INTRODUCTION

It is well known that a trilinear, binary form, $(\alpha r)(\beta s)(\gamma t)$ can be transformed into the canonical form $r_0 s_0 t_0 + r_1 s_1 t_1$ unless a certain invariant of the fourth degree vanishes. The vanishing of this invariant expresses that the two pairs $r_0 s_0$ which are neutral for t have come together. It is then at once expressible as the discriminant of a quadratic equation, i.e. as a two-rowed determinant whose elements are quadratic in the coefficients of the form. Due to the symmetry of the canonical form in the three digredient variables this invariant can be expressed in three ways as such a determinant. We wish to examine this triple expressibility, not in the relatively trivial case of a single trilinear form but rather for a net of such forms, i.e. for a quadrilinear form $A(1,1,1,2)$ the ternary variable being the parameter of the net.

§ 1

The general quadrilinear form, binary in variables r,s,t, and ternary in x, written symbolically as $A(1,1,1,2) = (\alpha r)(\beta s)(\gamma t)(\delta x)$ involves $(2\cdot2\cdot2\cdot3 - 1) - (3+3+3+8) = 6$ absolute constants. A curve of fourth degree in S_2 has $(4\cdot7)/2 - 8 = 6$ absolute constants also. The quartic curve R derived later from the form turns out to be a generic quartic, which, in turn, defines a finite number of such quadrilinear form.

A neutral triple $r_0 s_0 t_0$ of A with respect to x is defined to be values of r, s, and t which cause A to vanish identically in x. We expect a finite number of such neutral triples since there are three conditions - coefficients of x must all be zero - on the three degrees of freedom of r, s, and t.

Let us consider A as the trilinear, binary form $f(1,1,1) = (\alpha r)(\beta s)(\gamma t)$ whose coefficients $\alpha_i \beta_j \gamma_k$ $(i,j,k = 0,1)$ are linear in x. The concomitants of degree two in the coefficients of f are (*)

$$P_1 = (f,f)_{011} = (\beta\beta')(\gamma\gamma')(\alpha r)(\alpha' r)$$
$$P_2 = (f,f)_{101} = (\alpha\alpha')(\gamma\gamma')(\beta s)(\beta' s)$$
$$P_3 = (f,f)_{110} = (\alpha\alpha')(\beta\beta')(\gamma t)(\gamma' t).$$

The forms P_1, P_2, and P_3 when set equal to zero represent quadratic systems of conics, for, given a value of x, there are in general two values of r, s, and t which satisfy the respective equations. There is one concomitant of degree four in the coefficients of f which expresses the condition that P_3 be a perfect square. It is

$$R = (P_1, P_1)_{200} = (P_2, P_2)_{020} = (P_3, P_3)_{002} = (\alpha\alpha')(\beta\beta')(\gamma\gamma'')(\gamma'\gamma'')(\alpha''\alpha''')(\beta''\beta''').$$

R is an invariant of the form f and is of the fourth degree in x. This R is the quartic curve mentioned above. It may be expressed in terms of the coefficients of f as follows.

$$\begin{aligned} R = {} & a^1_{000} a^1_{111} + a^1_{001} a^1_{110} + a^1_{010} a^1_{101} + a^1_{100} a^1_{011} \\ & + 4a_{100} a_{010} a_{001} a_{111} + 4a_{000} a_{110} a_{101} a_{011} \\ & - 2a_{000} a_{111} a_{010} a_{101} - 2a_{000} a_{111} a_{100} a_{011} \\ & - 2a_{001} a_{110} a_{010} a_{101} - 2a_{001} a_{110} a_{100} a_{011} \\ & - 2a_{000} a_{111} a_{001} a_{110} - 2a_{010} a_{101} a_{100} a_{011} = 0. \end{aligned} \tag{1}$$

An interchange of subscripts leaves R unaltered showing that R = 0 is also the condition that P_2 and P_1 be perfect squares in s and r respectively. The conics of each of the above three systems touch the quartic at four points (**), they will be called contact systems henceforth. Thus we can state the

Theorem I: The quartic curve R as derived above from a quadrilinear form appears with three contact systems of conics individually isolated.

A contact system contains six conics each of which breaks up into two double tangents of the quartic curve. It is known (***) that pairs of contact systems are either syzygetic if they have four double tangents in common or azygetic if they have six double tangents in common. We proceed to show that the three systems above are such that any pair is azygetic.

§ 2

Consider the trilinear form $T(1,1,2) = (\beta s)(\gamma t)(\delta x)$ obtained from A for a fixed value of r. For a fixed $s = s^{(o)}$ and variable t this line $(\beta s^{(o)})(\gamma t)(\delta x) = 0$ turns around a point $x^{(o)}$, so that $s^{(o)} x^{(o)}$ is a pair neutral for t in $T(1,1,2) = 0$. It is shown that the equation of $x^{(o)}$ is

$$(\beta s^{(o)})(\beta' s^{(o)})(\gamma\gamma')(\delta\delta' x) = 0. \tag{2}$$

For the point $x^{(o)}$ the bilinear form in s and t, $(\beta s)(\gamma t)(\delta x^{(o)})$, has a neutral value $s = s^{(o)}$ and therefore breaks up into factors displaying the $s^{(o)}$ and a $t^{(o)}$. That is

$$(\beta s)(\gamma t)(\delta x^{(o)}) \equiv (s s^{(o)})(t t^{(o)}).$$

The condition that the bilinear form in s and t factor is that

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(*) Saddler, W. "Triple Binary Forms". Proceedings of the Cambridge Philosophical Society. Vol. 22, (1925), pp. 688 - 693.

(**) Salmon, G. "Higher Plane Curves". Third Edition, (1879), pp. 223 - 226.

(***) Weber, H. "Lehrbuch Der Algebra". Sweiter Band, (1896), pp. 357 - 366.

$$(3) \qquad (\beta\beta')(\gamma\gamma')(\delta x)(\delta' x) = 0.$$

Since (3) is symmetric with respect to the coefficients of r and s, then for point $x^{(o)}$ on this conic there is a value $t^{(o)}$ such that $t^{(o)}, x^{(o)}$ are neutral for s in $T(1,1,2)=0$, and the equation of the point $x^{(o)}$ in terms of $t^{(o)}$ is

$$(4) \qquad (\beta\beta')(\gamma t^{(o)})(\gamma' t^{(o)})(\delta\delta' \xi) = 0.$$

By symmetry in s and t, it is clear that the points $x^{(o)}$ in question lie on a single conic, hence

Theorem II: A point on the conic (3) is given parametrically by the equations (2), s being the parameter, and also by (4), t being the parameter. The form $T(1,1,2)=0$ yields for a given s and t the line joining the two points of this conic whose parameters are s and t.

It is shown that the equation of the line joining two points of conic (3) contains T as one factor. The other factor is the bilinear form $(\lambda s)(\mu t)$ which vanishes when the two points coincide. We prove

Theorem III: The linear transformation between the parameters s and t on conic (3) is given by the bilinear form $(\lambda s)(\mu t) = 0$. This is the bilinear form apolar to the net of bilinear forms given by variable x in $T = 0$.

The algebraic operation of joining the points (2) and (4) yields to within a numerical factor the result $(\rho s)(\gamma t)(\delta x)\ (\delta\delta'\delta'')(\beta\beta')(\gamma\gamma'')(\beta'' s)(\gamma' t)$ whence

$$(5) \qquad (\lambda s)(\mu t) = (\delta\delta'\delta'')(\beta\beta')(\gamma\gamma'')(\beta'' s)(\gamma' t).$$

If (5) degenerates, say $(\lambda s)(\mu t) = n(s s^{(1)})(t t^{(1)})$, it is then shown that $s^{(1)}, t^{(1)}$ is a pair neutral for x in T. It is also shown that conic (3) breaks up into two lines ξ_s with parameter $s^{(o)}$, and ξ_t with parameter $t^{(o)}$. We prove

Theorem IV: If $s^{(1)}, t^{(1)}$ is a neutral pair of $(\beta s)(\gamma t)(\delta x)$ then the conic (3) breaks up into a pair of lines whose equations are $\dfrac{(\beta s^{(1)})(\gamma t)(\delta x)}{(t t^{(1)})} = 0$, and $\dfrac{(\beta s)(\gamma t^{(1)})(\delta x)}{(s s^{(1)})} = 0$.

[A proper conic in the plane has a parametric equation $(a\xi)(\alpha t)^2 = 0$. This degenerates when a factor $(t t^{(o)})$ separates out leaving a factor which is the parametric equation of a single line and thus a line pair cannot be represented in the parametric form $(a\xi)(\alpha t)^2 = 0$. It is however of some interest to observe that the special (1,1,2) form with a neutral pair does represent a pair of lines with parameters s on one line and t on the other.]

§ 3

It was mentioned above that a finite number of neutral triads for A was expected. $(\lambda s)(\mu t)$ has coefficients which are of the third degree in the coefficients of $T(1,1,2)$, thus of the third degree in r. Since the condition that $(\lambda s)(\mu t)$ factor is of degree two in its coefficients, this condition is of degree six in the coefficients of A, whence

Theorem V: The quadrilinear form A has in general six triads $r^{(i)}, s^{(i)}, t^{(i)}$ $(i = 1, \cdots, 6)$ neutral for x.

§ 4

Consider one neutral triad $r^{(o)}, s^{(o)}, t^{(o)}$ of the quadrilinear form $(\alpha r)(\beta s)(\gamma t)(\delta x) = 0$. The conic defined by $r = r^{(o)}$ in the quadratic system, $(\alpha r)(\alpha' r)(\beta\beta')(\gamma\gamma')(\delta x)(\delta' x) = 0$ is a pair of lines and, according to Theorem IV, the equations of these lines are

$$(6) \qquad N_o \equiv \frac{(\alpha r^{(o)})(\beta s^{(o)})(\gamma t)(\delta x)}{(t t^{(o)})} = 0, \qquad M_o \equiv \frac{(\alpha r^{(o)})(\beta s)(\gamma t^{(o)})(\delta x)}{(s s^{(o)})} = 0.$$

The conic defined by $s = s^{(o)}$ in the quadratic system $(\alpha\alpha')(\beta s)(\beta' s)(\gamma\gamma')(\delta x)(\delta' x) = 0$ is also a pair of lines, and the equations of these lines are

$$(7) \qquad N_o \equiv \frac{(\alpha r^{(o)})(\beta s^{(o)})(\gamma t)(\delta x)}{(t t^{(o)})} = 0, \qquad L_o \equiv \frac{(\alpha r)(\beta s^{(o)})(\gamma t^{(o)})(\delta x)}{(r r^{(o)})} = 0.$$

Also the conic defined by $t = t^{(o)}$ in the quadratic system, $(\alpha\alpha')(\beta\beta')(\gamma t)(\gamma' t)(\delta x)(\delta' x) = 0$, is the pair of lines whose equations are

$$(8) \qquad M_o \equiv \frac{(\alpha r^{(o)})(\beta s)(\gamma t^{(o)})(\delta x)}{(s s^{(o)})} = 0, \qquad L_o \equiv \frac{(\alpha r)(\beta s^{(o)})(\gamma t^{(o)})(\delta x)}{(r r^{(o)})} = 0.$$

It is clear from (6), (7), and (8) that the neutral triad yields only three distinct lines, each line going with one of the three pairings of the three parameters in the triad. Thus each neutral triad $r^{(i)}, s^{(i)}, t^{(i)}$ will yield three lines L_i, M_i, and N_i. Moreover for the values $r^{(i)}$, $s^{(i)}$, or $t^{(i)}$ the six degenerate conics of their respective quadratic systems are $M_i N_i$, $N_i L_i$, or $L_i M_i$ respectively. Thus

Theorem VI: The three contact systems of Theorem I are such that any pair is azygetic.

§ 5

The six neutral triads $r^{(i)}, s^{(i)}, t^{(i)}$ contain the three binary sextics $r^{(1)}, \cdots, r^{(6)}$; $s^{(1)}, \cdots, s^{(6)}$; and $t^{(1)}, \cdots, t^{(6)}$ each on its individual binary domain. Each sextic has three absolute constants. However, the three sextics, being obtained from a form A with six absolute constants, must be subject to three mutual conditions. We show in this paragraph that two of the sextics may be chosen at random and the third is then uniquely determined to within a projectivity. We also show in the next paragraph that these three mutual conditions are satisfied by the unique determination of the sixth neutral triad when the first five neutral triads are given.

Consider a mapping from the plane x' which we will write as $[x']$, to the plane x which we will write as $[x]$, by a net of cubics on the seven points p_i $(i=1,\cdots,7)$ in $[x']$ (*). A point of $[x']$ is taken by this mapping into a single point of $[x]$; but a point of $[x]$ is mapped into two points of $[x']$. Thus the transformation followed by its inverse takes a point of $[x']$ into itself, and into another point of $[x']$. This is the well known Cremona involution, a transformation of degree eight.

There are twenty one composite cubics of the net which consist of a line on two of the points p_i and the conic on the remaining five points. There are also seven fundamental cubics of this net. They are the cubics having a double point at one of the points p_i and passing simply through the remaining six points.

We have shown that the jacobian of the net of cubics, which is a curve of order six and genus three, transforms into a quartic curve of $[x]$. We have also shown that the twenty eight bitangents of the quartic curve are transforms of the seven fundamental, and the twenty one composite cubics mentioned above.

The pencil of lines on p_1 corresponds under the Cremona involution to a pencil of quintics with a simple point at p_1 and double points at p_i $(i=2,\cdots,7)$, and conversely. For convenience we will write point p_i as (i). The sextics $(1,r)'(12^2,\cdots,7^2,r)^5$ form a quadratic system since they depend quadratically on the parameter r. It is shown that the map of this system is a contact system of conics of the quartic curve in $[x]$. The six lines of $(1,r)'$ which pass through the other six points p_i are given parameters $r^{(i)}$ $(i=1,\cdots,6)$. These six lines give rise to fixed sextics of the system of sextics. These fixed sextics break up into fundamental and composite cubics and thus transform into six pairs of bitangents of the quartic curve. Thus the six values $r^{(i)}$ are associated with six pairs of betangente which we call $D_{12}\cdot D_{18}, D_{13}\cdot D_{18}, \cdots, D_{17}\cdot D_{18}$. Similarly we establish the parameters $s^{(i)}$ $(i=1,\cdots,6)$ with the six lines of $(2,s)'$ which pass through p_i $(i=1,3,\cdots,7)$. Thus the six values $s^{(i)}$ are associated with the bitangents $D_{21}\cdot D_{28}$, $D_{23}\cdot D_{28}$, $\cdots$, $D_{27}\cdot D_{28}$.

The pencil of cubics $(1^2 34567,t)^3$ corresponds under the Cremona involution to the pencil of cubics $(2^2 34567,t)^3$ and conversely. The sextics $(1^2 34567,t)^3(2^2 34567,t)^3$ depend quadratically on the parameter t and transform into a third contact system of conics of the quartic curve in $[x]$. The cubic of $(1^2 34567,t)^3$ which passes through point p_2, and those five cubics which contain the lines $(1,3)'$, $(1,4)'$, $\cdots$, $(1,7)'$ are given the parameters $t^{(i)}$ $(i=1,\cdots,6)$. Thus the six values of $t^{(i)}$ are associated with the bitangents $D_{18}\cdot D_{26}, D_{13}\cdot D_{23}, \cdots, D_{17}\cdot D_{27}$. It is clear that these three quadratic systems of conics are azygetic.

Let there be given two pencils of lines in a plane, the six values $r^{(i)}$, and also the six values $s^{(i)}$. Let the common line have parameter $r^{(i)}$ in the first pencil and $s^{(i)}$ in the second pencil. Any other two lines in the first pencil are given the parameter $r^{(4)}$ and $r^{(5)}$, and two others in the second pencil are given parameters $s^{(4)}$ and $s^{(5)}$. Then three more lines of each pencil are determined as those whose cross ratios with respect to the three lines already selected equal the values of $r^{(j)}$ and $s^{(j)}$ $(j=4,5,6)$ respectively. These six lines determine seven points p_i $(i=1,\cdots,7)$. Since $t^{(i)}$ depend only on the seven points p_i we state the

Theorem VII: If the r and s values of each of the six neutral triads are chosen, the value of t in each triad is determined.

§ 6

In this paragraph we use a mapping(**) between the plane x and a net of quadric surfaces in S_3 to establish a correspondance between bitangents of a quartic curve in $[x]$, and composite quartics on the eight self associated points q_i $(i=1,\cdots,8)$. The composite quartics are of the type $(q_1,q_2)'(q_3,\cdots,q_8)^3$ where the line and twisted cubic are bisecant. There are twenty eight of these quartics. We consider three pencils of planes with vertices $(q_1,q_2)'$ $(q_1,q_8)'$, and $(q_2,q_8)'$. We have shown that the six planes of each pencil which pass through the other six points q_i give rise to pairs of bitangents of the quartic curve in $[x]$. It turns out that the three sets of six pairs of bitangents are such that any two sets are azygetic. Let r, s, and t be the parameters of the three pencils of planes such that

$q_1q_8q_2$	$q_2q_8q_1$	$q_1q_2q_8$	have parameters	$r^{(1)}s^{(1)}t^{(1)}$
$q_1q_8q_3$	$q_2q_8q_3$	$q_1q_2q_3$	" "	$r^{(2)}s^{(2)}t^{(2)}$
⋮	⋮	⋮	⋮ ⋮	⋮ ⋮ ⋮
$q_1q_8q_7$	$q_2q_8q_7$	$q_1q_2q_7$	" "	$r^{(6)}s^{(6)}t^{(6)}$

(*) de Paolis, R. "La trasformazione piana doppia di terzo ordine, primo genere, e la sua applicazione alle curve del quarto ordine". Atti Della Reale Accademie die Lincei. Memorie. Vol. 2, (1877-78). See also "Selected Topics in Algebraic Geometry". p. 125

(**) Snyder, V. and Sisam, C. H. "Analytic Geometry of Space" pp. 167-170.

Then if the base q_1, q_2, q_5, q_3, q_4 is chosen and the binary bases $r^{(1)}, r^{(2)}, r^{(3)}$; $s^{(1)}, s^{(2)}, s^{(3)}$; $t^{(1)}, t^{(2)}, t^{(3)}$ are also chosen, the points q_5 and q_6 are determined by the double ratios $r^{(4)} r^{(5)}$, $s^{(4)}, s^{(5)}$; $t^{(4)}, t^{(5)}$. Thus if the first five triads r, s, and t are chosen, the points $q_1, \cdots, q_6, q_7$ are determined. Point q_7 is the eighth base point of the net of quadric surfaces on these seven points and determines the sixth neutral triad $r^{(6)} s^{(6)} t^{(6)}$. We can then state the

Theorem VIII; If the first five neutral triads are chosen the sixth neutral triad is determined.

§ 7

It is shown in this paragraph that a trilinear, binary form can be interpreted as a cubic surface in S_3. This is done by choosing three skew lines in S_3, say λ, μ, and ν as vertices of three pencils of planes with parameters r, s, and t. When r, s, and t satisfy the equation $(\alpha r)(\beta s)(\gamma t) = 0$, the point common to the three planes describes the cubic surface. It is shown that this cubic surface contains the lines λ, μ, and ν.

Four triads which satisfy the form determine four points on the cubic surface. Through the three skew lines and the four points there is a web of cubic surfaces. A fifth arbitrary triad yields a net of cubic surfaces which can be represented by a form A, x being the parameter of the net. This form has a sixth neutral triad, i.e. the net of cubic surfaces is on a sixth point and the correspondance between the fifth and sixth point is a Cremona involution in space (*). Thus Theorem VIII is a statement of a property of this Cremona involution.

8

We have seen in the last paragraph that in general $(\alpha r)(\beta s)(\gamma t) = 0$ can be represented as a cubic surface. Let us select the lines λ, μ, and ν to form a triangle with vertices q_1, q_2, and q_3. The cubic surface will have nodes at q_1, q_2, and q_3. The plane $q_1 q_2 q_3$ has parameters $r^{(1)}$, $s^{(1)}$, and $t^{(1)}$ in the respective pencils of planes. If $(\alpha r^{(1)})(\beta s^{(1)})(\gamma t^{(1)}) = 0$, all points of this plane are on the cubic surface, thus plane $q_1 q_2 q_3$ factors out leaving a quadric with simple points at q_1, q_2, and q_3. If also $(\alpha r^{(j)})(\beta s^{(j)})(\gamma t^{(j)}) = 0$ for $(j = 2, \cdots, 5)$, then there is a net of quadrics given by a net of (1,1,1) forms $(\alpha r)(\beta s)(\gamma t)(\delta x)$ (x_0, x_1, and x_2 being the parameters of the net), on q_1, q_2, and q_3 and also on the points $q_3, \cdots, q_6$ resulting from the triads $r^{(j)}$, $s^{(j)}$, and $t^{(j)}$ $(j = 2, \cdots, 5)$. The net of quadrics is on an eighth point given by $r^{(6)}, s^{(6)}$, and $t^{(6)}$, which is the sixth neutral triad of $(\alpha r)(\beta s)(\gamma t)(\delta x)$. Hence the quadrilinear form A with which we began can be interpreted as a net of quadrics on a prescribed set of base points and thus the generic quartic surve $R = 0$ which determines 36 such nets of quadrics (**), determines $36 \cdot \binom{8}{3} = 2016$ such quadrilinear forms A.

(*) Coble, A. B. "Groups of Cremona Transformations in Space of Planar Type II". Duke Mathematical Journal. Vol. 2, (1936), pp. 205-219.

(**) Hesse, O. "Uber die Doppeltangenten der Curven vierter Ordnung". Journal fur die reine und angewandte Mathematik. Vol. 49-50, (1855), pp. 279-332.

VITA

Charles Frederick Strobel was born April 22, 1911, in Buffalo, New York. He attended the public schools of Buffalo and was graduated from Technical High School in June, 1929. From then until the fall of 1931 he was employed by the Western Electric Company. During these two years he attended the evening sessions of Hutchinson Central High School to earn college entrance requirements. From September, 1931, until June, 1935, he attended the University of Buffalo where he received the A.B. degree (cum laude). He was an assistant at the University of Buffalo from 1935 until June, 1937, when he received the A.M. degree. The title of the master's thesis is "Pencils of Conics and Conditions for Relatedness." He spent the next four academic years at the University of Illinois as an assistant in the department of mathematics.

He is a member of Phi Eta, the American Mathematical Association, the American Mathematical Society, and the Society of Sigma Xi.

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